The Kepler Elements and the PQW Frame

It is necessary to specify an set of initial conditions to completely determine the trajectory of an orbit. Since Newton's Law of gravity

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{\mu\hat{r}}{r^3}$$

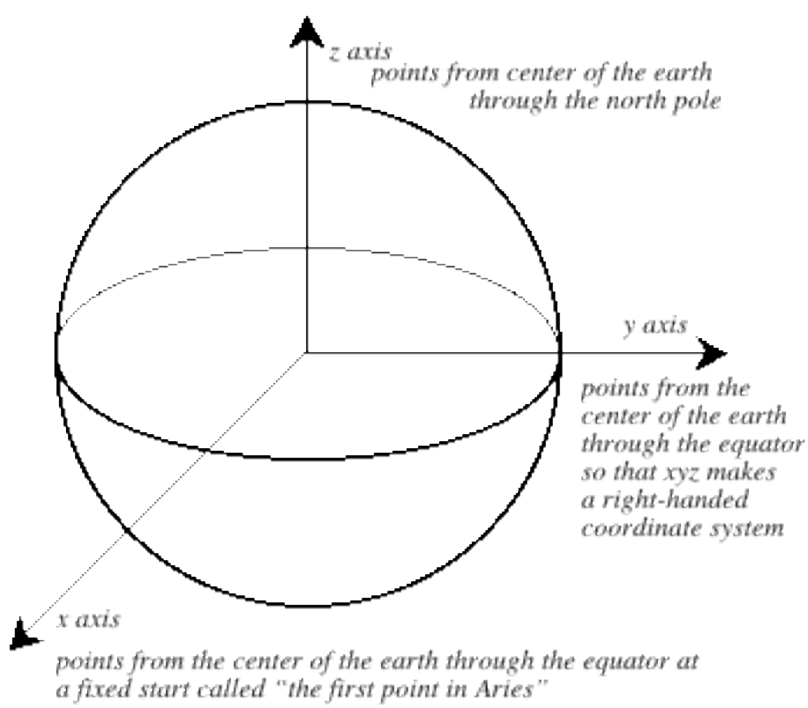
is a second order differential equation in the position, we need to specify both the position and velocity (i.e., both $\vec{r}(0)$ and $\vec{r}\,'(0) = \vec{v}(0)$). Since there are three dimensions, we need to specify a total of six parameters, i.e.,

$$x_0, y_0, z_0, v_{x,0}, v_{y,0}, v_{z,0}$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position and $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ is the velocity (the subscripts on the velocity refer to the components and are not partial derivatives). Alternatively, it is possible to specify six other numbers that give the same information; for example, we might give the information in polar coordinates.

It turns out that there is a very natural description of the orbit within the orbit plane, that is given in terms of the three parameters a (the semimajor axis), e (the eccentricty), and θ (the true anomaly). Furthermore, we can specify the orientation of the plane of the orbit using three additional angles that we will define shortly.

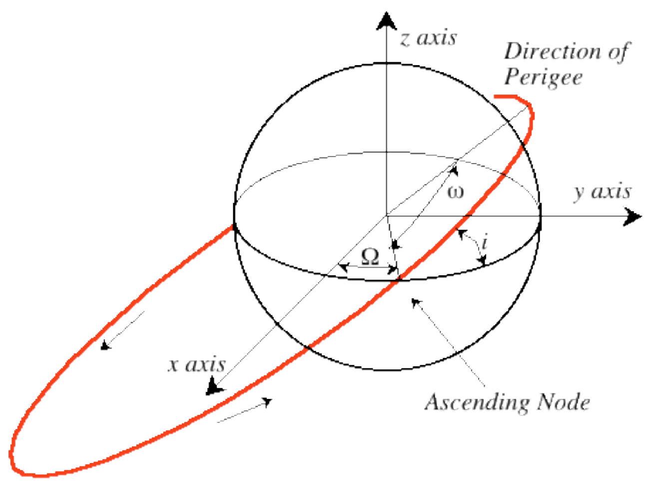
The so called "Earth Centered Inertial" coordinate frame is illustrated in the following figure.



In this coordinate system the x-axis points at a fixed spot in space called the ***first point in Aries,*** and sometimes called the ***Vernal Equinox***. Technically, the vernal equinox is a point in time, and not a point in space. The reason for the this confusion is that the x-axis points from the center of the earth to the sun at the time of the vernal equinox. At this point in time, the sun is entering the constellation Aires (or it least it was some 3000 years ago when the constellations were named, it actually drifts with a period of around 24,000 years). It is usually easier to just think of the x-axis as pointing to a fixed star (which really doesn't exist but we know where, or at least in what direction, it should be).

These axes (of the XYZ frame) are fixed in the center of the Earth but their directions are fixed in space. That means that every point on the Earth (excepting the poles) rotate 360˚ around the z-axis every day! Furthermore, although the directions of the axes are fixed, the center of the coordinate system is not, as it moves around the sun every 365 days. Thus the XYZ system is not really inertial ("fixed" in space) but is a moving coordinate system. For this reason it is called an ***Earth Centered Inertial*** coordinate frame – it is sort of fixed – so long as we stay very close to the Earth. When we are dealing with Earth orbiting satellites virtually no accuracy at all is lost by treating this coordinate system as truly fixed in space; however, when studying interplanetary trajectories, we must account for the Earth's motion.

The plane of the orbit intersects the Earth's equator at an angle i, called the ***inclination*** of the orbit.



The orientation of the orbit with respect to the XYZ frame is defined in terms of three angles:

i, the ***inclination***, the angle between the Earth's equation and the plane of the orbit.

Ω, the ***right ascension of ascending node***, the angle measured in the equator between the x axis and the intersection of the satellite's orbit and equatorial plane as it moves from the southern hemisphere to the northern hemisphere. This point of intersection is called the ***ascending node*** of the orbit.

ω, the ***argument of perigee***, the angle measured in the plane of the orbit between the ascending node and the perigee.

We can specify the initial conditions in terms of the following set of elements:

$(a,\ e,\ i,\ \Omega,\ \omega,\ \theta)$

More commonly the mean anomaly is used instead of the true anomaly:

$(a,\ e,\ i,\ \Omega,\ \omega,\ M)$

Either of these sets of elements are called the **Kepler** (or **Keplerian**) **Orbital Elements**.

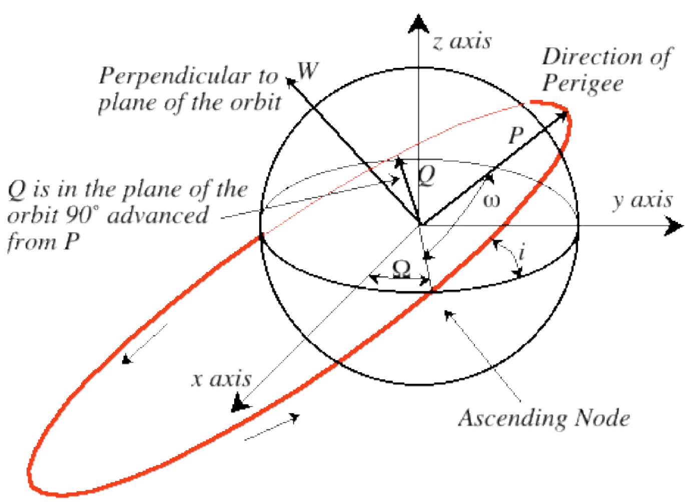
A new frame, called the PQW frame is defined in terms of these angles, as follows:

$\vec{P}$ points from the center of the Earth towards the direction of perigee, in the plane of the orbit.

$\vec{Q}$ lies in the plane of the orbit and points from the center of Earth in a direction 90° advanced from the direction of perigee.

$\vec{W}$ points from the center of the Earth in a direction perpendicular to the plane of the orbit in such a way that the triad $\vec{P},\vec{Q},\vec{W}$ forms a right-handed coordinate system, i.e., $\vec{W} = \vec{P} \times \vec{Q}$

This set of coordinate axes is illustrated in the following figure.



Although they are drawn in the figure with different lengths, **the vectors** $\vec{P},\vec{Q},\vec{W}$ **are all unit vectors.** We can use the angles shown in the figure to determine the components of the vectors $\vec{P},\vec{Q},\vec{W}$ in terms of the vectors $\vec{i},\vec{j},\vec{k}$.

The easiest way to derive the coordinate transformation is via rotation matrices. To rotate the *x* axis onto the *P* axis, while at the same time rotating the *y* axis onto the *Q* axis, and the *z* axis on the *W* axis, we would perform the following three rotations, in succession:

(1) Rotate by Ω about the *z* axis.

(2) Rotate by i about the new x axis.

(3) Rotate by ω about the newer z axis.

$$\begin{pmatrix} \vec{P} \\ \vec{Q} \\ \vec{W} \end{pmatrix} = R_z(\omega) R_x(i) R_z(\Omega) \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix} \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\cos i \sin\Omega & \cos i \cos\Omega & \sin i \\ \sin i \sin\Omega & -\sin i \cos\Omega & \cos i \end{pmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$$

Should be $\sin i \cos\omega$

$$= \begin{pmatrix} \cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega & \cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega & \sin\omega\sin i \\ -\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega & -\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega & \sout{\sin i\cos\Omega} \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{pmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$$

Reading off each component, we have

$P_x = \cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega$

$P_y = \cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega$

$P_z = \sin\omega\sin i$

$Q_x = -\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega$

$Q_y = -\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega$

$Q_z = \sout{\sin i\cos\Omega}$

$W_x = \sin i\sin\Omega$

$W_y = -\sin i\cos\Omega$

$W_z = \cos i$